

# The Representation of Numbers by States in Quantum Mechanics

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February 1, 2008

## Abstract

The representation of numbers by tensor product states of composite quantum systems is examined. Consideration is limited to  $k$ -ary representations of length  $L$  and arithmetic mod  $k^L$ . An abstract representation on an  $L$  fold tensor product Hilbert space  $\mathcal{H}^{arith}$  of number states and operators for the basic arithmetic operations is described. Unitary maps onto a physical parameter based tensor product space  $\mathcal{H}^{phy}$  are defined and the relations between these two spaces and the dependence of algorithm dynamics on the unitary maps is discussed. The important condition of efficient implementation by physically realizable Hamiltonians of the basic arithmetic operations is also discussed.

## 1 INTRODUCTION

The representation of numbers by states of physical systems is basic and widespread in science. However, this representation is assumed and used implicitly, with little effort devoted to exactly what assumptions and conditions are implied. Here this question is examined for the nonnegative integers. Consideration will be limited to  $k$ -ary representations of length  $L$  of numbers by tensor product states and to arithmetic modulo  $k^L$ .

Based on the universality of quantum mechanics, all physical systems of interest are quantum systems described by pure or mixed quantum states. This is the case whether the systems are microscopic, as is the case for quantum computers, or macroscopic, as is the case for all presently existing computers. Microscopic systems are those for which the ratio  $t_{dec}/t_{sw} \gg 1$  where  $t_{dec}$  and  $t_{sw}$  are the decoherence and switching times [1]. In this case the system remains isolated from the environment for a time duration of many switching steps. If  $t_{dec}/t_{sw} < 1$  then the system is macroscopic, and environmental interactions stabilize some system states (the pointer states) [2] for a time duration of many

switching steps. The emphasis here is on microscopic systems although much of the material also holds for macroscopic systems.

One route to the exact meaning of the representation of numbers by states of physical systems begins with pure mathematics. A nonempty set is a set of numbers if and only if it satisfies (i.e. is a model of) the axioms of number theory or arithmetic [3]. Here these axioms need to be modified by inclusion of relevant axioms for a commutative ring with identity as these axioms are satisfied by modular arithmetic [4]. Details of the axioms are not important here. However, necessary conditions for a nonempty set to be a model include the existence of functions or operators with the properties of basic arithmetic operations, the successor  $S$  (or  $+1$ ),  $+$ , and  $\times$ , given by the axioms. The ordering relation and the induction schema will not be discussed here.

If the axioms are consistent then there are many mathematical models of the axioms. Included are models containing tensor product states and unitary operators on a product Hilbert space of states. A model on an abstract product Hilbert space  $\mathcal{H}^{arith}$  is described in Section 2.

The connection to physics is made in two steps. First a Hilbert space  $\mathcal{H}^{phy}$  based on two sets of physical parameters is described. Models, based on  $\mathcal{H}^{phy}$ , of the axioms are described in Section 3. These are generated by unitary operators from  $\mathcal{H}^{arith}$  to  $\mathcal{H}^{phy}$ .

The second step is the requirement that there exist physical models of the axioms in which the basic arithmetic operations are efficiently implementable. The widespread existence of computers shows that this existence requirement is satisfied, at least for macroscopic systems.

This requirement is discussed in Section 4 and applied to models based on  $\mathcal{H}^{phy}$ . Due to space limitations the discussion in this and other sections is brief. Details and proofs are provided elsewhere [5].

## 2 Models Based on $\mathcal{H}^{arith}$

Let  $\mathcal{H}^{arith} = \otimes_{j=1}^L \mathcal{H}_j$  where  $\mathcal{H}_j$  is a  $k$  dimensional Hilbert space spanned by states  $|h, j\rangle$  where  $0 \leq h \leq k-1$  and  $j$  is fixed. States in the corresponding basis spanning  $\mathcal{H}^{arith}$  have the form  $|\underline{s}\rangle = \otimes_{j=1}^L |\underline{s}(j), j\rangle$  where  $\underline{s}$  is any function from  $1, \dots, L$  to  $0, \dots, k-1$ . The value of  $j$  distinguishes the component states (or qubytes) and  $h$  ranges over the  $k$  possible values of the states for each component.

The reason  $j$  is part of the state and not a subscript, as in  $\otimes_{j=1}^L |\underline{s}(j)\rangle_j$ , is that the action of operators to be defined depends on the value of  $j$ . Expression of this dependence is not possible if  $j$  appears as a subscript and not between  $|$  and  $\rangle$ .

Definitions of  $S$ ,  $+$ ,  $\times$  are required by the axioms. The efficient implementation requirement necessitates the definition of  $L$  different successor operators,  $V_j^{+1}$ , for  $j = 1, \dots, L$ . These operators are defined to correspond to the addition of  $k^{j-1} \bmod k^L$  where  $V_1^{+1}$  corresponds to  $S$ .

To define the  $V_j^{+1}$  let  $u_j$  be a cyclic shift of period  $k$  that acts on the states  $|h, j\rangle$  according to  $u_j|h, j\rangle = |h+1 \bmod k, j\rangle$ .  $u_j$  is the identity on all states  $|m, j'\rangle$  where  $j' \neq j$ . Define  $V_j^{+1}$  by

$$V_j^{+1} = \sum_{n=j}^L u_n P_{(\neq k-1), n} \prod_{\ell=j}^{n-1} u_\ell P_{(k-1), \ell} + \prod_{\ell=j}^L u_\ell P_{(k-1), \ell} \quad (1)$$

Here  $P_{(k-1), j} = |k-1, j\rangle\langle k-1, j| \otimes 1_{\neq j}$  is the projection operator for finding the  $j$  component state  $|k-1, j\rangle$  and the other components in any state.  $P_{m, j}$  and  $u_j$  satisfy the commutation relation  $u_j P_{m, j} = P_{m+1, j} u_j \bmod k$  for  $m = 0, \dots, k-1$ . Also  $P_{(\neq k-1), j} = 1 - P_{(k-1), j}$ . In this equation the unordered product is used because for any  $p, q$ ,  $u_m P_{p, m}$  commutes with  $u_n P_{q, n}$  for  $m \neq n$ . Also for  $n = j$  the product factor with  $j \leq \ell \leq n-1$  equals 1.

The operator  $+$  is defined on  $\mathcal{H}^{arith} \otimes \mathcal{H}^{arith}$  by  $|\underline{s}\rangle|\underline{w}\rangle = |\underline{s}\rangle|\underline{s+w}\rangle$  where

$$|\underline{s+w}\rangle = V_L^{+\underline{s}(L)} V_{L-1}^{+\underline{s}(L-1)} \dots V_1^{+\underline{s}(1)} |\underline{w}\rangle \quad (2)$$

and  $V_j^{+\underline{s}(j)} = (V_j^{+1})^{\underline{s}(j)}$ .

Note that for pairs of product states, which are first introduced here, the domains of the functions  $\underline{s}$  and  $\underline{w}$  must be different. That is  $|\underline{s}\rangle|\underline{w}\rangle = |\underline{s * w}\rangle$  where  $*$  denotes concatenation and  $\underline{s * w}$  is a function from  $1, \dots, 2L$  to  $0, \dots, k-1$ . This follows from the requirement that all components of  $|\underline{w}\rangle$  must be distinguished from those in  $|\underline{s}\rangle$ .

There are some basic properties the operators  $V_j^{+1}$  must have: they are cyclic shifts on  $\mathcal{H}^{arith}$  and they satisfy

$$V_{j+1}^{+1} = (V_j^{+1})^k \text{ for } 1 \leq j < L: (V_L^{+1})^k = 1 \quad (3)$$

This shows the exponential dependence on  $j$  and the need for separate definitions and efficient implementation of each of these operators. Also both the  $V_j^{+1}$  and  $+$  are unitary. Proofs of these and other properties and a definition of  $\times$  are given elsewhere [5].

The proof that the operators  $V_j^{+1}$ ,  $+$ ,  $\times$  and states of the form  $|\underline{s}\rangle$  in  $\mathcal{H}^{arith}$  are a model of modular arithmetic consists in showing that the appropriate axioms are satisfied. Some details of this are given in [5]. Note that  $\mathcal{H}^{arith}$  is also a model of the Hilbert space axioms.

### 3 Models Based on $\mathcal{H}^{phy}$

The model of modular arithmetic described so far is abstract. No connection to quantum physics is provided. To remedy this one needs to describe models based on physical parameters. Let  $A$  and  $B$  be sets of  $L$  and  $k$  physical parameters for physical observables  $\hat{A}$ ,  $\hat{B}$  for an  $L$  component quantum system. The

parameters in  $A$  distinguish the  $L$  component systems from one another and the parameters of  $B$  refer to  $k$  different internal physical states of the component systems. Examples of  $A$  include a set of locations in space or a set of excitation energies, as is used in NMR quantum computers [6]. Examples of  $B$  include spin projections along an axis or energy levels of a particle in a potential well.

The physical parameter based Hilbert space  $\mathcal{H}^{phy} = \otimes_{a \in A} \mathcal{H}_a$  where  $\mathcal{H}_a$  is spanned by states of the form  $|b, a\rangle$  with  $b \in B$  and  $a$  fixed.  $\mathcal{H}^{phy}$  is spanned by states  $|\underline{t}\rangle = \otimes_{a \in A} |\underline{t}(a), a\rangle$  where  $\underline{t}$  is a function from  $A$  to  $B$ .

The presence of the  $a$  component in the state  $|\underline{t}(a), a\rangle$  property in the state is essential in that the state of the composite quantum system contains all the quantum information available to any physical process or algorithm. It is used by algorithms to distinguish the different components or qubits. This is especially the case for any algorithm whose dynamics is described by a Hamiltonian that is selfadjoint and time independent. This is an example of Landauer's dictum "Information is Physical" [7].

The goal here is for physical parameter states such as  $|\underline{t}\rangle$  to represent numbers. However, it is clear that the product states  $|\underline{t}\rangle = \otimes_{a \in A} |\underline{t}(a), a\rangle$  do not represent numbers. The reason is that there is no association between the labels  $a$  and powers of  $k$ ; also there is no association between the range set  $B$  of  $\underline{t}$  and numbers.

This can be remedied by use of unitary maps from  $\mathcal{H}^{arith}$  to  $\mathcal{H}^{phy}$  that preserve the tensor product structure. Let  $g$  and  $d$  be bijections (one-one onto) maps from  $1, \dots, L$  to  $A$  and from  $0, \dots, k-1$  to  $B$ . For each  $g, d$ , and  $j$ ,  $w_{g,d,j}$  is a unitary operator that maps states  $|h, j\rangle$  in  $\mathcal{H}_j$  to states in  $\mathcal{H}_{g(j)}$  according to  $w_{g,d,j}|h, j\rangle = |d(h), g(j)\rangle$ . This induces a unitary operator  $W_{g,d} = \otimes_{j=1}^L w_{g,d,j}$  from  $\mathcal{H}^{arith}$  to  $\mathcal{H}^{phy}$  where

$$\begin{aligned} W_{g,d}|\underline{s}\rangle &= \otimes_{j=1}^L w_{g,d,j}|\underline{s}(j), j\rangle \\ &= \otimes_{j=1}^L |d(\underline{s}(j)), g(j)\rangle = |\underline{s}_g^d\rangle. \end{aligned} \quad (4)$$

Here  $|\underline{s}_g^d\rangle$  is the physical parameter based state in  $\mathcal{H}^{phy}$  that corresponds, under  $W_{g,d}$  to the number state  $|\underline{s}\rangle$  in  $\mathcal{H}^{arith}$ .

Conversely the adjoint operator  $W_{g,d}^\dagger$  relates physical parameter states in  $\mathcal{H}^{phy}$  to number states in  $\mathcal{H}^{arith}$ :

$$\begin{aligned} W_{g,d}^\dagger|\underline{t}\rangle &= \otimes_{a \in A} w_{g^{-1}, d^{-1}, a}|\underline{t}(a), a\rangle \\ &= \otimes_{a \in A} |d^{-1}(\underline{t}(a)), g^{-1}(a)\rangle = |\underline{t}_{g^{-1}}^{d^{-1}}\rangle. \end{aligned} \quad (5)$$

Here  $|\underline{t}_{g^{-1}}^{d^{-1}}\rangle$  is the number state in  $\mathcal{H}^{arith}$  that corresponds to the physical state  $|\underline{t}\rangle$ . Note that  $W_{g,d}^\dagger = W_{g^{-1}, d^{-1}}$  where  $g^{-1}, d^{-1}$  are the inverses of  $g$  and  $d$ , and  $w_{g^{-1}, d^{-1}, a} = w_{g, d, g^{-1}(a)}^\dagger$ .

The operators  $W_{g,d}$  also induce representations of the  $V_j^{+1}$ ,  $+$ , and  $\times$  operators on the physical parameter states. For the  $V_j^{+1}$  one defines  $V_{g,j}^{d,+1}$  by

$$V_{g,j}^{d,+1} = W_{g,d} V_j^{+1} W_{g,d}^\dagger. \quad (6)$$

An equivalent definition can be obtained from Eq. 1 by replacing  $u_\ell$  by  $u_{g(\ell)}^d = w_{g,d,\ell} u_\ell w_{g,d,\ell}^\dagger$  and  $P_{k-1,\ell}$  by  $P_{d(k-1),g(\ell)} = w_{g,d,\ell} P_{k-1,\ell} w_{g,d,\ell}^\dagger$ .

In a similar fashion the  $W_{g,d}$  are used to define  $+_{g,d}$  acting on the physical parameter states in  $\mathcal{H}^{phy} \otimes \mathcal{H}^{phy}$ . The operator  $+_{g,d}$  is defined in terms of the operator  $+$  on  $\mathcal{H}^{arith} \otimes \mathcal{H}^{arith}$  by  $+_{g,d} = (W_{g,d} \otimes W_{g,d}) + (W_{g,d}^\dagger \otimes W_{g,d}^\dagger)$ .  $\times_{g,d}$  is defined similarly.

There are a large number of tensor product preserving unitary maps from  $\mathcal{H}^{arith}$  to  $\mathcal{H}^{phy}$ . Each of these induces a model for the axioms of modular arithmetic on  $\mathcal{H}^{phy}$ . There are  $L!k!$  of these maps restricted to the form given by Eq. 4 for  $W_{g,d}$  as there are  $L!$  bijections  $g$  and  $k!$  bijections  $d$ . Thus there is no unique correspondence between number states  $|\underline{s}\rangle$  in  $\mathcal{H}^{arith}$  and physical parameter states  $|\underline{t}\rangle$  in  $\mathcal{H}^{phy}$ . In general the physical state  $|\underline{s}_g^d\rangle$  corresponding to the number state  $|\underline{s}\rangle$  depends on  $g$  and  $d$ . Conversely, by Eq. 5, the number state  $|\underline{t}_{g-1}^{d-1}\rangle$  corresponding to the physical state  $|\underline{t}\rangle$  in  $\mathcal{H}^{phy}$  depends on  $g$  and  $d$ .

The question arises regarding the dependence of the dynamics of a quantum algorithm on the  $W_{g,d}$ . Some algorithms are independent of these maps; others are not. In general, since the the dynamics of any algorithm is physical, it must be described on  $\mathcal{H}^{phy}$ . If the algorithm can also be defined on  $\mathcal{H}^{phy}$ , then the dynamics is independent of these maps. Grover's Algorithm [8] is an example of this type of algorithm as it can be described and implemented on states in  $\mathcal{H}^{phy}$  that are linear superpositions of  $|\underline{t}\rangle$ . No reference to states in  $\mathcal{H}^{arith}$  is needed.

However, arithmetic algorithms must be defined on  $\mathcal{H}^{arith}$ . For these the dynamics does depend on these maps. Shor's Algorithm [9] is an example of this as it describes the computation of a numerical function  $f_m(x) = m^x \bmod M$  where  $M$  and  $m$  are relatively prime.

So far the discussion has been limited to models of modular arithmetic on  $\mathcal{H}^{arith}$  and on  $\mathcal{H}^{phy}$ . However the full connection to physics has not yet been established. This is given by the condition of efficient implementation of the basic arithmetic operations.

## 4 Efficient Implementability of the Basic Arithmetic Operations

The meaning of this important requirement is that it must be possible to physically implement the basic operations and that the implementation must be efficient. That is, for the  $V_{g,j}^{d,+1}$ , for each  $j$  there must exist a physically realizable Hamiltonian  $H_{g,j}^d$  and a time  $t_j$  such that

$$e^{-iH_{g,j}^d t_j} = V_{g,j}^{d,+1}. \quad (7)$$

This definition is quite general in that the Hamiltonian can depend on  $j$ . However for many systems and dynamics a Hamiltonian  $H_g^d$  that implements  $V_{g,j}^{d,+1}$  and is independent of  $j$  is realizable.

The requirement of efficiency means that both the space-time and the thermodynamic resources required to physically implement the operations must be at most polynomial in  $L, k$ . This condition excludes  $k = 1$  or unary representations as all arithmetic operations are exponentially hard in this case. Large values of  $k$  are also excluded as distinguishing among a large set of symbols and carrying out simple arithmetic operations becomes thermodynamically expensive. Also there are physical limitations on the amount of information that can be reliably stored and distinguished per unit space time volume [10].

Thermodynamic resources are needed to protect the system from errors resulting from operation in a noisy environment. Microscopic systems also need protection from decoherence [11]. Methods include the use of quantum error correction codes [12], EPR pairs [13], and decoherence free subspaces [14]. Protection of macroscopic systems is less difficult since one takes advantage of decoherence to give stabilized "pointer" [2] states that represent numbers in a macroscopic computer.

The reason for separate definitions of the  $V_{g,j}^{d,+1}$  for each  $j$  is that the requirement means that each of these operators for  $j = 1, \dots, L$  must be efficiently implemented. If the  $V_{g,j}^{d,+1}$  were defined in terms of iterations of  $V_{g,1}^{d,+1}$ , then implementation of  $V_{g,j}^{d,+1}$  would require  $k^{j-1}$  iterations of  $V_{g,1}^{d,+1}$ . This is not efficient even if  $V_{g,1}^{d,+1}$  can be efficiently implemented.

Since the  $V_{g,j}^{d,+1}$ ,  $+_{g,d}$ , and  $\times_{g,d}$  operators are many system nonlocal operators, many dynamical steps would be needed to implement these operators by a realizable two particle local Hamiltonian, Eq. 7. As is well known, there are many methods of efficiently implementing these operators, at least in macroscopic computers. For the  $V_{g,j}^{d,+1}$  methods include moving the procedure for efficiently implementing  $V_{g,1}^{d,+1}$  along the path  $g$  in  $A$  to the site  $g(j)$ . For  $+_{g,d}$ , methods are based on iterations of the  $V_{g,j}^{d,+1}$ , Eq. 2.

The thermodynamic resources required to physically implement the  $V_{g,j}^{d,+1}$  and other arithmetic operations also depend on the path  $g$ . In general paths are chosen that respect the topological or neighborhood properties of  $A$  as this reduces the resources required. But in principle any path is possible as the resource dependence on the path choice is not exponential.

## 5 Discussion

The importance of the efficient implementability condition must be emphasized. Besides excluding  $k = 1$  and large values of  $k$ , it excludes most unitary maps from  $\mathcal{H}^{arith}$  to  $\mathcal{H}^{phy}$ . To see this one notes that for any unitary  $U$ , tensor product preserving or not, if the states  $|\underline{s}\rangle$  and operators  $V_j^{+1}$ ,  $+$ ,  $\times$  satisfy the axioms of modular arithmetic, so do the states  $U|\underline{s}\rangle$  and the operators  $UV_j^{+1}U^\dagger$ ,  $(U \otimes U) + (U^\dagger \otimes U^\dagger)$  and an operator for  $\times$ . However, for most  $U$ , these operators are not efficiently implementable. Also the states  $U|\underline{s}\rangle$  may not be efficiently preparable. This is the main reason the  $U$  were taken to have the form  $W_{g,d}$ .

Much remains to be done. Future work includes dropping the modulo limitation and considering other types of numbers. The use of annihilation and creation operators to represent states needs examination. Also the exact meaning of physical realizability needs to be clarified.

## acknowledgments

This work is supported by the U.S. Department of Energy, Nuclear Physics Division, under contract W-31-109-ENG-38.

## References

- [1] DiVincenzo, D. P. Science **270**: 255, (1995); Los Alamos Archives quant-ph/**0002077**.
- [2] W. H. Zurek, Phys. Rev. D **24**: 1516, (1981); **26**: 1862 (1982); E. Joos and H. D. Zeh, Z. Phys. B **59**: 23, (1985); H. D. Zeh quant-ph/**9905004**; E Joos, quant-ph/**9808008**.
- [3] J. R. Shoenfield, *Mathematical Logic* (Addison-Wesley, Reading, MA 1967); R. Smullyan, *Gödel's Incompleteness Theorems* (Oxford University Press, Oxford, 1992).
- [4] I. T. Adamson, *Introduction to Field Theory*, 2nd. Edition, Cambridge University Press, London, 1982.
- [5] P. Benioff, Los Alamos Archives Preprint quant-ph/**0003063**.
- [6] N.A. Gershenfeld, Science **275**: 350 (1997); D.G. Cory, A.F. Fahmy, and T.F. Havel, Proc. Natl. Acad. Sci. **94**: 1634 (1997).
- [7] R. Landauer, Physics Today **44**: No 5, 23, (1991); Physics Letters A **217**: 188, (1996); in *Feynman and Computation, Exploring the Limits of Computers*, A.J.G.Hey, Ed., (Perseus Books, Reading MA, 1998).
- [8] L.K.Grover, in *Proceedings of 28th Annual ACM Symposium on Theory of Computing* ACM Press New York 1996, p. 212; Phys. Rev. Letters, **79**: 325 (1997); Phys. Rev. Letters, **80**: 4329 (1998).
- [9] P. W. Shor, in *Proceedings, 35th Annual Symposium on the Foundations of Computer Science*, S. Goldwasser (Ed), IEEE Computer Society Press, Los Alamitos, CA, 1994, pp 124-134; SIAM J. Computing, **26**: 1481 (1997).
- [10] S. Lloyd, Los Alamos Archives preprint quant-ph/9908043; Y. J. Ng, Los Alamos Archives Preprint quant-ph/0006105.
- [11] W. H. Zurek, Physics Today **44**: No. 10, 36 (1991); J.R. Anglin, J. Paz, and W. H. Zurek, Phys. Rev A **55**: 4041 (1997); W. G. Unruh, Phys. Rev. A **51**: 992 (1995).

- [12] R. Laflamme, C. Miquel, J. P. Paz, and W. H. Zurek, Phys. Rev. Letters **77**: 198 (1996); D. P. DiVincenzo and P. W. Shor, Phys. Rev. Letters **77**: 3260 (1996); E. M. Rains, R. H. Hardin, P. W. Shor, and N. J. A. Sloane, Phys. Rev. Letters **79**: 954 (1997); E. Knill, R. Laflamme, and W. H. Zurek, Science **279**: 342 (1998).
- [13] C.H. Bennett in *Feynman and Computation, Exploring the Limits of Computers* A.J.G.Hey, Ed., (Perseus Books, Reading MA, 1998); C.H.Bennett D.P.DiVincenzo, C.A.Fuchs, T.Mor, E.Rains, P.W.Shor, J.A.Smolín, and W.K.Wooters, Rev. A **59**: 1070 (1999).
- [14] D. Bacon, D. A. Lidar and K. B. Whaley, Phys.Rev. **A60**: (1999) 1944.